1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
Aerospace Medical Research Laboratory	Unclassified
Aerospace Medical Div, Air Force Systems Command	2b. GROUP
Wright-Patterson Air Force Base, Ohio 45433	N/A

3. REPORT TITLE

A MATHEMATICAL MODEL FOR THE BEHAVIOR OF THE BRAIN WHEN THE HUMAN HEAD IS SUBJECTED TO IMPULSIVE LOADS.

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

Ali E. Engin, Ph.D. and Verne L. Roberts, Ph.D.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 1971	28 27	8

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO. 7231	AMRL-TR-71-29
	Paper No. 36.
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Aerospace Medical Research Laboratory
	Aerospace Medical Div, Air Force Systems
	Command, Wright-Patterson AFB, OH 45433

13. ABSTRACT

The Symposium on Biodynamics Models and Their Applications took place in Dayton, Ohio, on 26-28 October 1970 under the sponsorship of the National Academy of Sciences - National Research Council, Committee on Hearing, Bioacoustics, and Biomechanics; the National Aeronautics and Space Administration; and the Aerospace Medical Research Laboratory, Aerospace Medical Division, United States Air Force. Most technical areas discussed included application of biodynamic models for the establishment of environmental exposure limits, models for interpretation of animal, dummy, and operational experiments, mechanical characterization of living tissue and isolated organs, models to describe man's response to impact, blast, and acoustic energy, and performance in biodynamic environments.

DD FORM 1473 1 NOV 65

Security Classification

A MATHEMATICAL MODEL FOR THE BEHAVIOR OF THE BRAIN
WHEN THE HUMAN HEAD IS SUBJECTED TO IMPULSIVE LOADS

by

Ali E. Engin, Ph.D.
and
Verne L. Roberts, Ph.D.
The University of Michigan

ABSTRACT

The subject matter of this paper is concerned with the theoretical determination of the behavior of the brain when the human head is subjected to external impulsive loads. The mathematical analysis is made for the axisymmetric response of an inviscid compressible fluid loaded impulsively by its elastic spherical shell container. The motion of the fluid is assumed to be governed by the linear wave equation. The spherical shell equations include both membrane and bending effects in axisymmetric torsionless motion. In the analysis first the solution for an initial value problem is obrained; later the transient response of the fluid for an arbitrary velocity input of the shell is constructed by means of convolution integral. For the numerical results, a characteristic time is defined and the excess pressure distribution in the fluid is evaluated for various deceleration times comparable with this characteristic time. A description of some of the salient features of the excess pressure distribution is also given in view of the elastic and rigid shell boundary of the fluid. Since the problem is considered from a theoretical point of view to determine some of the causes of the brain damage when the human head is subjected to impulsive loads, in the numerical computations the data is chosen to be suitable to the physical properties of brain and skull.

LIST OF SYMBOLS

E Young's modulus

$P_n(\cos \phi)$ Legendre polynomials of the first kind

$P_n^1(\cos \phi)$ Associated Legendre polynomials of the first kind and first order

V Speed of the fluid-filled shell along an axis passing through the poles

Φ Velocity potential for the fluid

Φ_1 Nondimensional velocity potential for the fluid, Φ/ac_s

Ω Nondimensional frequency, $\omega a/c$

a Radius of spherical shell

a_0, a_n Coefficients of Legendre polynomial expansion of ζ

b_n Coefficients of Legendre polynomial expansion of ψ

c_0, c_n Coefficients of velocity potential

c Compressional wave speed in the fluid

c_s Apparent wave speed in the shell, $[E/\rho_s(1-\nu^2)]^{1/2}$

f Shell-fluid parameter, $\rho_0 a/\rho_s h$

h Shell thickness

$j_n(z)$ Spherical Bessel function, $(\pi/2z)^{1/2}J_{n+1/2}(z)$

k Wave number, ω/c

p Excess pressure

p_1 Nondimensional excess pressure, $p/\rho_0 c_s^2$

r, θ, ϕ Spherical coordinates

r_1 Nondimensional radius, r/a

s Speed ratio, c/c_s

t Time

t_c Characteristic time, $\frac{2a}{c}$

t_0 Deceleration time

u Meridional displacement of the shell mid-surface with respect to geometric center of the shell

w Radial displacement of the shell mid-surface with respect to geometric center of the shell

α^2 Thickness parameter, $h^2/12a^2$

ζ Nondimensional radial displacement, w/a

ψ Nondimensional meridional displacement, u/a

λ_n $n(n+1)$

ν Poisson's ratio

ρ_0, ρ_s Mass density of fluid and shell respectively

τ Nondimensional time, $c_s t/a$

τ_c Nondimensional characteristic time, $2/s$

τ_0 Nondimensional deceleration time, $c_s t_0/a$

ω Angular frequency

INTRODUCTION

This investigation aims at the following two considerations. First, the subject matter is a point of interest in theoretical mechanics due to the fluid-solid interaction nature of the problem. Second, from the application point of view, the impulsive response of the fluid when the enclosing elastic shell is suddenly subjected to a change in its velocity can be taken as a simple but improved theoretical model to determine the formation of brain damage when no local contacts are made on the human head.

The previous studies in the area have been either investigations involving shells in contact externally and/or internally with fluids, especially by researchers in the field of acoustics, or analyses of various head injury models. While studies on these two categories are numerous, only a few representative ones will be mentioned here. Junger[1] investigated the effect of fluid on the natural frequencies of cylindrical and spherical shells freely suspended in a compressible fluid medium. Free and forced oscillations of infinitely long cylindrical shells surrounded by water were studied by Greenspon[2] who treated unpressurized shells by exact elasticity theory and cylindrical shells with fluid by approximate shell theory. Utilizing linear shell theory, which includes both membrane and bending effects, Engin and Liu[3] recently obtained the frequency equation and corresponding frequency spectrum of fluid-filled spherical shells for the axisymmetric and nontorsional motion.

In the literature, the rigorous mathematical treatments of the physical theory of the formation of brain damage was first introduced by Anzelius[4] and Güttinger[5] with their analyses of the impulsive response of an inviscid fluid contained in a rigid closed spherical shell (or container). Their formulations are essentially identical and involve an axisymmetric solution of the acoustic wave equation in

880

spherical coordinates. In the papers of both authors the eigen values of the problem are determined by requiring the radial component of the fluid velocity to vanish at the interior surface of the rigid spherical shell surrounding the fluid. Hayashi[6] treated a one-dimensional version of the Anzelius-Güttinger model. His model consists of a rigid and massless vessel containing inviscid fluid. The vessel, which is attached to a linear spring, is subjected to impacts with a stationary wall. Approximate solutions were obtained for the limiting cases of soft and very hard impacts. Although this simple model has the advantage of being easy to interpret, it has the similar shortcomings of the Anzelius-Güttinger model, i.e. (a) due to rigidity and geometric assumption there is no way to determine the possible locations of skull fracture and (b) the effects of skull deformation on the intracranial pressure distribution can not be determined. Recently, Engin[7] removed some major restrictions of previous head injury models by obtaining analytical and numerical solutions for the dynamic response of a fluid-filled elastic spherical shell. The loading pattern for his model is taken to be local, radial, axisymmetric and impulsive. Since the load is applied as a force locally on one of the poles of the shell, the combined shell theory which includes both membrane and bending effects of the shell has been used for the proper description of the wave propagation on the shell. The analysis utilizes Laplace transform technique in obtaining the transient response of the system. The conclusions of Engin's paper include the possible locations of brain damage and skull injury on the basis of the numerical computations. As a problem in mechanics, the present investigation is a generalization of the results of Anzelius and Güttinger by removing the restriction of rigidity of the shell surrounding the fluid. Our model consists of an elastic spherical shell filled with inviscid compressible fluid. The shell material and fluid are considered to be homogeneous and isotropic. In the analysis, the fluid-filled shell will be considered to have a

constant translational velocity for t<o with respect to an inertial coordinate system. At t=o the shell is brought to a sudden stop, i.e. the fluid occupying the interior space of the shell is subjected to a global axisymmetric impulse on its boundary. The determination of the pressure distribution in the fluid for this kind of impulse will help to explain quantitatively the location and the magnitude of brain damage under the conditions in which the application of local forces on the skull is avoided.

I. EQUATIONS OF MOTION AND THEIR SOLUTIONS

The governing differential equations of motion for a fluid-filled spherical shell were previously obtained in reference 3 by means of Hamilton's Principle. These three partial differential equations, which are coupled in terms of the meridional and radial displacements, u, w of the shell mid-surface and velocity potential, ϕ, of the fluid, are given below in nondimensional form:

$$\alpha^2\left[\frac{\partial^2\psi}{\partial\phi^2} + \cot\phi\,\frac{\partial\psi}{\partial\phi} - (\nu+\cot^2\phi)\psi - \frac{\partial^3\zeta}{\partial\phi^3} - \cot\phi\,\frac{\partial^2\zeta}{\partial\phi^2} + (\nu+\cot^2\phi)\frac{\partial\zeta}{\partial\phi}\right] + \frac{\partial^2\psi}{\partial\phi^2} + \cot\phi\,\frac{\partial\psi}{\partial\phi}$$

$$- (\nu+\cot^2\phi)\psi + (1+\nu)\frac{\partial\zeta}{\partial\phi} - \frac{\partial^2\psi}{\partial\tau^2} = 0, \tag{1}$$

$$\alpha^2\left[\frac{\partial^3\psi}{\partial\phi^3} + 2\cot\phi\,\frac{\partial^2\psi}{\partial\phi^2} - (1+\nu+\cot^2\phi)\frac{\partial\psi}{\partial\phi} + (\cot^2\phi-\nu+2)\psi\cot\phi - \frac{\partial^4\zeta}{\partial\phi^4} - 2\cot\phi\,\frac{\partial^3\zeta}{\partial\phi^3}\right.$$

$$\left.+ (1+\nu+\cot^2\phi)\frac{\partial^2\zeta}{\partial\phi^2} - (2-\nu+\cot^2\phi)\cot\phi\,\frac{\partial\zeta}{\partial\phi}\right] - (1+\nu)\left(\frac{\partial\psi}{\partial\phi} + \psi\cot\phi + 2\zeta\right) - \frac{\partial^2\zeta}{\partial\tau^2}$$

$$- f\,\frac{\partial\phi_1(1,\phi,\tau)}{\partial\tau} = 0, \tag{2}$$

and

$$\frac{1}{r_1^2}\frac{\partial}{\partial r_1}\left(r_1^2\,\frac{\partial\phi_1}{\partial r_1}\right) + \frac{1}{r_1^2\sin\phi}\,\frac{\partial}{\partial\phi}\left(\sin\phi\,\frac{\partial\phi_1}{\partial\phi}\right) - \frac{1}{s^2}\frac{\partial^2\phi_1}{\partial\tau^2} = 0 \tag{3}$$

where

$$\psi = \frac{u}{a}, \quad \zeta = \frac{w}{a}, \quad \tau = \frac{c_s}{a}t, \quad c_s = \left[E/\rho_s(1-\nu^2)\right]^{1/2}, \quad s = \frac{c}{c_s}, \quad r_1 = \frac{r}{a}, \quad \phi_1 = \frac{\phi}{ac_s},$$

$$f = \frac{\rho_0 a}{\rho_s h}.$$

Since the deformation of a given shell is usually analyzed in terms of the deformation of its mid-surface, Eqs. 1 and 2 describe the motion of an imaginary boundary of the fluid which is half the shell thickness away from the physical boundary. These shell equations include both membrane and bending effects and they are given for an axisymmetric torsionless motion. Equation 3 is the linear

883

wave equation describing the motion of small oscillations of inviscid and ir-rotational fluid.

Consider the following series expansions for the nondimensional radial and tangential displacements of the shell midsurface:

$$\zeta(\phi,\tau) = \sum_{n=0}^{\infty} a_n(\tau) P_n(\cos\phi), \qquad (4a)$$

and

$$\psi(\phi,\tau) = \sum_{n=1}^{\infty} b_n(\tau) P_n'(\cos\phi), \qquad (4b)$$

where $P_n(\cos\phi)$ are Legendre polynomials of the first kind and $P_n'(\cos\phi)$ are asso-ciated Legendre polynomials of the first order, first kind. Since the second solutions of the Legendre equations are singular at the poles they are not in-cluded in the expansions of ζ and ψ. The requirement of boundedness of solutions and the linearity of Eq. 3 lead to its formal solution:

$$\phi_i(r_i,\phi,\tau) = \sum_{n=0}^{\infty} c_n(\tau) j_n(kar_i) P_n(\cos\phi), \qquad (5)$$

where $j_n(kar_i)$ is spherical Bessel function, $k=\omega/c$ is the wave number and ω is the circular frequency.

The boundary condition between the fluid and shell can be stated as the continuity of normal velocities for all ϕ and τ; that is,

$$\frac{\partial\zeta(\phi,\tau)}{\partial\tau} = \frac{\partial\phi_i(1,\phi,\tau)}{\partial r_i} \qquad (6)$$

Substitution of Eqs. 4a and 5 into Eq. 6 yields the following relationship between $a_n(\tau)$ and $c_n(\tau)$ for each n:

$$c_n(\tau) = \frac{1}{ka j_n'(ka)} \frac{da_n(\tau)}{d\tau} \qquad (7)$$

It can be shown that substitution of Eqs. 4a, 4b, and Eq. 5, together with Eq. 7 into Eqs. 1 and 2, and the repeated utilization of the differential equations satisfied by P_n and P_n' yields the following system of equations for the determination of $a_n(\tau)$ and $b_n(\tau)$:

for n=o:

$$\left[1 + f \frac{J_0(\Omega)}{\Omega J_0'(\Omega)}\right] \frac{d^2 a_0(\tau)}{d\tau^2} + 2(1+\nu) \, a_0(\tau) = 0, \tag{8}$$

for $n \geq 1$:

$$\frac{d^2 b_n(\tau)}{d\tau^2} + [1+\nu-\alpha^2(1-\nu-\lambda_n)]a_n(\tau) - (1-\nu-\lambda_n)(1+\alpha^2)b_n(\tau) = 0, \tag{9}$$

$$\left[1 + f \frac{J_n(\Omega)}{\Omega J_n'(\Omega)}\right] \frac{d^2 a_n(\tau)}{d\tau^2} - \left\{(1+\nu)\lambda_n + \alpha^2 \left[\lambda_n^2 - \lambda_n(1-\nu)\right]\right\} b_n(\tau) + \left\{2(1+\nu)\right.$$

$$\left. + \alpha^2[\lambda_n^2 - \lambda_n(1-\nu)]\right\} a_n(\tau) = 0 \tag{10}$$

where $\Omega = ka = \frac{\omega a}{c}$ and $\lambda_n = n(n+1)$. In reference 3 a description of some of the salient features of the frequency spectrum of a fluid-filled spherical shell is given in detail. There it was also shown that for each mode number, n, the infinite number of frequencies (or characteristic roots) exists. Thus, the solutions of Eqs. 8, 9 and 10 can be written in the following form:

$$a_0(\tau) = \sum_{m=0}^{\infty} A_{om} \sin(\Omega_{om} \, s\tau + \alpha_{om}), \tag{11}$$

$$a_n(\tau) = \sum_{m=0}^{\infty} A_{nm} \sin(\Omega_{nm} \, s\tau + \alpha_{nm}) \tag{12}$$

and

$$b_n(\tau) = \sum_{m=0}^{\infty} \xi_{nm} \, A_{nm} \sin(\Omega_{nm} \, s\tau + \alpha_{nm}) \tag{13}$$

where A_{om}, A_{nm}, δ_{nm} and α_{nm} are arbitrary constants; Ω_{om} and Ω_{nm} are the roots of the following frequency equations respectively

$$\left[1 + f\frac{J_0(\Omega)}{\Omega J_0^{'}(\Omega)}\right] s^2\Omega^2 - 2(1+\nu) = 0 \qquad \text{for } n=0, \tag{14}$$

and

$$\left[1 + f\frac{J_n(\Omega)}{\Omega J_n^{'}(\Omega)}\right] s^4\Omega^4 + \left\{[1+f\frac{J_n(\Omega)}{\Omega J_n^{'}(\Omega)}](1-\nu-\lambda_n)(1+\alpha^2)-2(1+\nu)-\alpha^2[\lambda_n^2-\lambda_n(1-\nu)]\right\} s^2\Omega^2$$

$$- (1+\nu)\left\{2(1-\nu-\lambda_n)(1+\alpha^2) + \lambda_n[1+\nu-\alpha^2(1-\nu-\lambda_n)]\right\} -\alpha^2(2-\lambda_n)[\lambda_n^2-\lambda_n(1-\nu)] = 0$$

for $n \geq 1$. $\tag{15}$

One of the steps in derivation of Eq. 15 also yields the following expression for δ_{nm}:

$$\delta_{nm} = \frac{1+\nu-\alpha^2(1-\nu-\lambda_n)}{s^2\Omega_{nm}^2+(1-\nu-\lambda_n)(1+\alpha^2)} \qquad \begin{array}{l} n=1,2,3,\dots \\ m=0,1,2,\dots \end{array}$$

In view of Eqs. 7, 11 and 12 the nondimensional velocity potential, ϕ, now can be written as

$$\phi_i(r_i,\phi,\tau) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{nm} s \frac{J_n(\Omega_{nm}r_i)}{J_n^{'}(\Omega_{nm})} \cos(\Omega_{nm} s\tau+\alpha_{nm}) P_n(\cos\phi) \tag{16}$$

Next, let us consider the following physical situation in which a fluid-filled shell is travelling with a constant speed, V, along an axis passing through the poles. Assume at t=o it is brought to a sudden stop. Incidentally, this situation can be visualized as the motion of the fluid-filled shell in a force field which can only affect the shell material; the parameters of the force field can be adjusted in such a manner that the desired motion of the shell is obtained. Due to inertia of the fluid particles, at time t=o they will experience a velocity relative to the coordinate system whose origin is located at the geometric center of the shell. In other words, when the shell surrounding the fluid is brought to a sudden stop at t=o, the fluid particles occupying the interior space of the

shell are unaware that the motion of shell is arrested. Assuming that the initial excess pressure distribution in the fluid is zero we can now write the following initial conditions on the nondimensional velocity potential ϕ_1

$$\phi_1(r_1,\phi,\tau=0)\equiv\phi_0=\frac{V}{c_s}\ r_1\ \cos\phi$$

(17)

and

$$\frac{\partial\phi_1}{\partial\tau}\bigg|_{\tau=0}=\phi_0'=0$$

From the second initial condition, i.e. $\phi_0'=0$, we get $\alpha_{nm}=0$. The first initial condition is used to obtain the coefficients A_{nm}. In order to do this let us first write Eq. 16 in the terminology of eigenfunctions by defining $e_{nm}=j_n(\Omega_{nm}r_1)P_n(\cos\phi)$:

$$\phi_1(r_1,\phi,\tau)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}s\ A_{nm}\ e_{nm}\ \frac{\cos(\Omega_{nm}\ s\tau)}{j_n'(\Omega_{nm})}$$

(18)

Let $\bar{e}_{nm}=j_n(\Omega_{n1}r_1)P_n(\cos\phi)$ be the conjugate-eigenfunction for e_{nm}; having applied the first of the initial conditions on ϕ_1 in Eq. 18 and multiplying both sides of the resulting equation by $\bar{e}_{nm}$ and integrating it over the fluid volume we get

$$A_{nm}=\frac{\int_\Psi\phi_0\ \bar{e}_{nm}\ dv}{\int_\Psi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{s}{j_n'(\Omega)}\ \bar{e}_{nm}\ e_{nm}\ dv}\ .$$

(19)

To evaluate the integral on the numerator of A_{nm} we write $\cos\phi$ as $P_1(\cos\phi)$ and see that in view of orthogonality of Legendre polynomials in the interval $[-1,1]$

$$\int_\Psi\phi_0\ \bar{e}_{nm}\ dv=\int_0^1\int_0^\pi\int_0^{2\pi}\frac{V}{c_s}(ar_1)^3 j_n(\Omega_{n1}r_1)P_1(\cos\phi)P_n(\cos\phi)\ \sin\phi\,d\phi\,d\theta\,dr_1=0$$

for $n\neq 1$.

(20)

Expressing the spherical Bessel functions in terms of fractional order Bessel functions and performing the integration in Eq. 20 yields

$$\int_V \phi_0 \, \bar{e}_{nm} \, dv = \frac{4\pi}{3} \frac{V}{c_s} \frac{a^3}{\Omega_m^{3/2}} J_{5/2}(\Omega_m)$$ (21)

where $\Omega_m \equiv \Omega_{1m}$. For the evaluation of the integral on the denominator of A_{nm} we use the following properties of Bessel functions and Legendre polynomials:

$$\int_0^a J_{n+\frac{1}{2}}(k_{n1}r) \, J_{n+\frac{1}{2}}(k_{nm}r) r \, dr = \begin{cases} \frac{1}{2}a^2 \left[J_{n+\frac{1}{2}}^2(k_{nm}a) - J_{n-\frac{1}{2}}(k_{nm}a)J_{n+\frac{3}{2}}(k_{nm}a) \right] & 1=m \\ \\ 0 & , \ 1 \neq m \end{cases}$$

and from the definition of the norm of $P_n(\cos\phi)$

$$\int_0^\pi P_n^2(\cos\phi) \sin\phi \, d\phi = \frac{2}{2n+1} \quad .$$

Thus, the value of the integral under consideration becomes

$$\int_V \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{s}{J_n^2(\Omega)} \bar{e}_{nm} e_{nm} \, dv = \frac{2\pi a^3 s}{2n+1} \frac{\left[J_{n+\frac{1}{2}}^2(\Omega_{nm}) - J_{n-\frac{1}{2}}(\Omega_{nm}) J_{n+\frac{3}{2}}(\Omega_{nm}) \right]}{\Omega_{nm}^{1/2} \left[\frac{n}{\Omega_{nm}} J_{n+\frac{1}{2}}(\Omega_{nm}) - J_{n+\frac{3}{2}}(\Omega_{nm}) \right]}$$ (22)

From Eq. 20 we see that $A_{nm}=0$ except $n=1$. Let $A_m \equiv A_{1m}$, using Eq. 21 and 22 in Eq. 19 we obtain the final expression for A_m:

$$A_m = \frac{2V \, J_{\frac{5}{2}}(\Omega_m) \left[\frac{1}{\Omega_m} J_{\frac{3}{2}}(\Omega_m) - J_{\frac{5}{2}}(\Omega_m) \right]}{c\Omega_m \left[J_{\frac{3}{2}}^2(\Omega_m) - J_{\frac{1}{2}}(\Omega_m) J_{\frac{5}{2}}(\Omega_m) \right]}$$ (23)

where Ω_m are the roots of the frequency equation for $n=1$. Substitution of Eq. 23 into Eq. 18 after some simplification gives the expression for the nondimensional velocity potential for the fluid

$$\phi_1(r_1, \phi, \tau) = \frac{2V}{c_s} \cos\phi \sum_{m=0}^\infty \frac{J_{\frac{5}{2}}(\Omega_m) \, J_{\frac{3}{2}}(\Omega_m r_1) \cos(\Omega_m s\tau)}{\Omega_m \left[J_{\frac{3}{2}}^2(\Omega_m) - J_{\frac{1}{2}}(\Omega_m) J_{\frac{5}{2}}(\Omega_m) \right] r_1^{1/2}}$$ (24)

The nondimensional excess pressure, p_1, is equal to $-\frac{\partial \phi_1}{\partial \tau}$ and it is obtainable directly from Eq. 24.

It is interesting to note that the appropriate limiting case of Eq. 24 agrees with the result obtained by Güttinger[5]. When we write Eq. 24 in dimensional form, c_s, the wave speed on the shell, disappears from the expression of velocity potential. In the case in which the shell surrounding the fluid becomes rigid, $s \to 0$, and Eqs. 14 and 15 degenerate to $j_n'(\Omega)=0$, which is easily shown to be the frequency equation of an ideal fluid contained in a rigid spherical shell. When the shell becomes rigid $\Omega_m's$ in Eq. 24 are taken to be the roots of $j_1'(\Omega)=0$ which can also be written in terms of fractional order Bessel functions as $2\Omega J_{\frac{3}{2}}'(\Omega)-J_{\frac{3}{2}}(\Omega)=0$. This last equation was used by Güttinger to obtain the necessary natural frequencies of the fluid for the determination of the impulsive response.

If desired, the transient response of the fluid for an arbitrary velocity input, $V(\tau)$, of the surrounding shell can be examined by means of Convolution integral. By the principle of superposition, it can be shown that velocity potential $\tilde{\phi}_1$, to an arbitrary shell excitation $V(\tau)$ can be expressed as

$$\tilde{\phi}_1(r_1,\phi,\tau) = \frac{V(0)}{V} \phi_1(r_1,\phi,\tau) + \int_0^\tau \frac{V'(\xi)}{V} \phi_1(r_1,\phi,\tau-\xi)d\xi, \quad \text{for } \tau \ge 0. \tag{25}$$

where $\phi_1(r_1,\phi,\tau)$ is given in Eq. 24 and V is the constant speed before the sudden stop of the spherical shell.

II. NUMERICAL RFSULTS AND DISCUSSION

For the determination of the numerical values of the excess pressure, p_i, generated in the fluid we use a different form of Eq. 24 in which the Bessel functions of fractional order are expressed in terms of trigonometric functions. The expression, thus obtained for p_i is

$$p_i(r_i,\phi,\tau) = \frac{2Vc}{c_s^2} \cos\phi \sum_{m=0}^{\infty} \frac{[3(\sin\Omega_m - \cos\Omega_m) - \Omega_m\sin\Omega_m][\sin(\Omega_m r_i) - \Omega_m r_i \cos(\Omega_m r_i)]\sin(\Omega_m s\tau)}{r_i^2[\Omega_m^2 + (1-3\Omega_m)\sin^2\Omega_m + \Omega_m\sin\Omega_m\cos\Omega_m]}$$

(26)

Here the dimensional excess pressure, p, can be obtained from $p=p_i\rho_e c_s^2$. At $r_i = o$ the equation for p_i is indeterminate; however, application of L'Hospital rule to Eq. 26 once yields $p_i(o,\phi,\tau)=o$. That is, in the equitorial plane which is perpendicular to the direction of impulse, pressure is zero at all times. Since the problem was considered to serve as a theoretical model to determine the formation of brain damage, the numerical values are obtained from the following data suitable to the physical properties of brain and skull:

$\rho_s = 0.0772$ lbm/in^3

$E = 2 \times 10^6$ lbf/in^2

$\nu = 0.25$

$a = 3$ in   (27)

$h = 0.15$ in

$\rho_e = 0.0362$ lbm/in^3

$c = 57100$ in/sec.

The assumptions leading to the above data are discussed in detail in references 7 and 8.

Since the sudden stop of a fluid-filled shell demands an infinite deceleration, it is more reasonable to consider the case where stopping occurs during a finite time t_0 (Nondimensional value, $\tau_0 = \frac{ct_0}{a}$). More specifically let us assume that the shell has the following velocity form

$$
V(t) = \begin{cases} V & (t \leq 0) \\ Vt/t_0 & (0 \leq t \leq t_0) \\ 0 & (t \geq t_0) \end{cases} \tag{28}
$$

i.e. it is brought to a stop with a constant deceleration during time t_0. For the numerical results the magnitude of V is taken to be 528 in/sec. As regards the response of the system, the above situation is identical to that in which the shell and its content are set into motion from rest with the same magnitude of acceleration. Thus, in view of Eq. 25 the expression for the excess pressure can be obtained directly from Eq. 26

$$
p_1(r_1,\phi,\tau) = \sum_{m=0}^{\infty} g_m(\phi,r_1) \left\{ [u(\tau)-u(\tau-\tau_0)]\sin^2 \frac{\Omega_m s\tau}{2} \right.
$$

$$
\left. + u(\tau-\tau_0) \sin \frac{\Omega_m s\tau_0}{2} \sin\Omega_m s(\tau-\frac{\tau_0}{2}) \right\} \tag{29}
$$

where

$$
g_m(\phi,r_1) = \frac{4V}{c_s\tau_0} \frac{[3(\sin\Omega_m-\cos\Omega_m)-\Omega_m\sin\Omega_m][\sin(\Omega_m r_1)-\Omega_m r_1\cos(\Omega_m r_1)]}{r_1^2\Omega_m[\Omega_m^2+(1-3\Omega_m)\sin^2\Omega_m+\Omega_m\sin\Omega_m\cos\Omega_m]} \cos\phi,
$$

and $u(\tau)$, $u(\tau-\tau_0)$ are the unit step functions. We note that the limiting case of Eq. 29 in fact gives Eq. 26 when $\tau_0 \to 0$. The series in Eq. 29 exhibits much better convergence than the one given in Eq. 26.

Let us define a characteristic time, t_c (or $\tau_c = \dfrac{ct_c}{a}$, 3.615 for the data

given in Eq. 27) to be the time required for a wave in the fluid to travel from

one pole to the other. In all the figures the deceleration duration is considered

with respect to this characteristic time. Figure 1 shows the plot of the excess

pressure at the pole ($\phi=o$) for two different values of deceleration duration. The

reduction of the pressure with doubling of the deceleration time is readily apparent.

Figures 2 through 8 present the excess pressure distribution along the polar axis

for various values of nondimensional radius r, and nondimensional time τ. From

Eq. 29 we see that $p_i(-r_i,\phi,\tau) = -p_i(r_i,\phi,\tau)$; hence, the points located symmetri-

cally with respect to the equitorial plane which is perpendicular to the polar

axis always experience the excess pressure of opposite sign. That is, for such

a pair of points if one is in a state of compression, the other one will be in

a state of dilatation. From Figs. 2 through 8 one can find the magnitude of the

excess pressure along any ray extending from the origin by simply multiplying the

value with $\cos\phi$. In other words, the waves generated on the inner surface of the

shell have amplitude factors which attenuate according to the polar angle ϕ. When

the deceleration time $\tau_0 \ll \tau_c$, sharp extremes in the excess pressure form for $\tau > \tau_0$.

This fact is illustrated in Fig. 2. A further interesting observation is the

comparison of the pressure distributions in the fluid contained in the rigid and

the elastic shells. In Figs. 3 through 8 pressure distributions are plotted for

both elastic and rigid cases. From these plots we see that the amplitude of the

excess pressure wave in the fluid contained in an elastic shell is considerably

less than that of the excess pressure wave in the fluid contained in a rigid shell.

Local variations of the wave form in the elastic case are also noticeable. When

we compare Figs. 5 and 7 and Figs. 6 and 8 we notice that the wave forms are

essentially the same and the amplitudes of pressure for the large value of τ_0 are much less than those corresponding to a small value of τ_0 (compare the scales in the figures). In Figs. 7 and 8 $\tau_0 = 100\tau_c$ corresponds to 502 in/sec^2 (or 130 g's for a deceleration time of 0.0105 sec.) In this case the maximum negative excess pressure generated in the fluid is about 48 psi.

In conclusion, if we seek the brain damage to occur at the points of rarefaction of the fluid, we find this situation arises in maximum magnitudes at the poles and these locations are quite significant for the analytical confirmation of the cavitation theory of brain damage.

ACKNOWLEDGEMENT
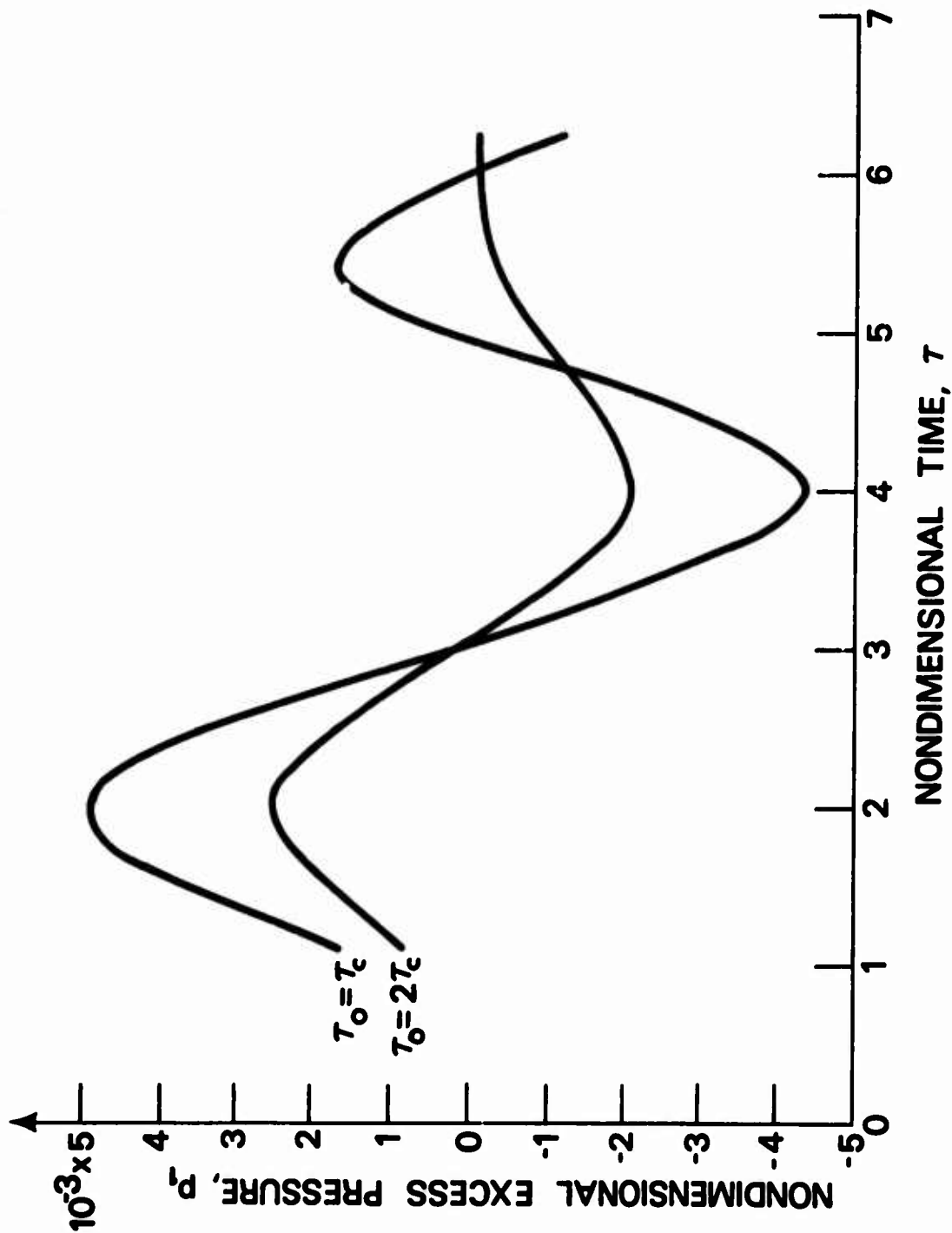
Fig. 1 Nondimensional excess pressure, $p_{_1}$, vs. nondimensional time, τ, at $\phi=0$, (Elastic shell boundary).
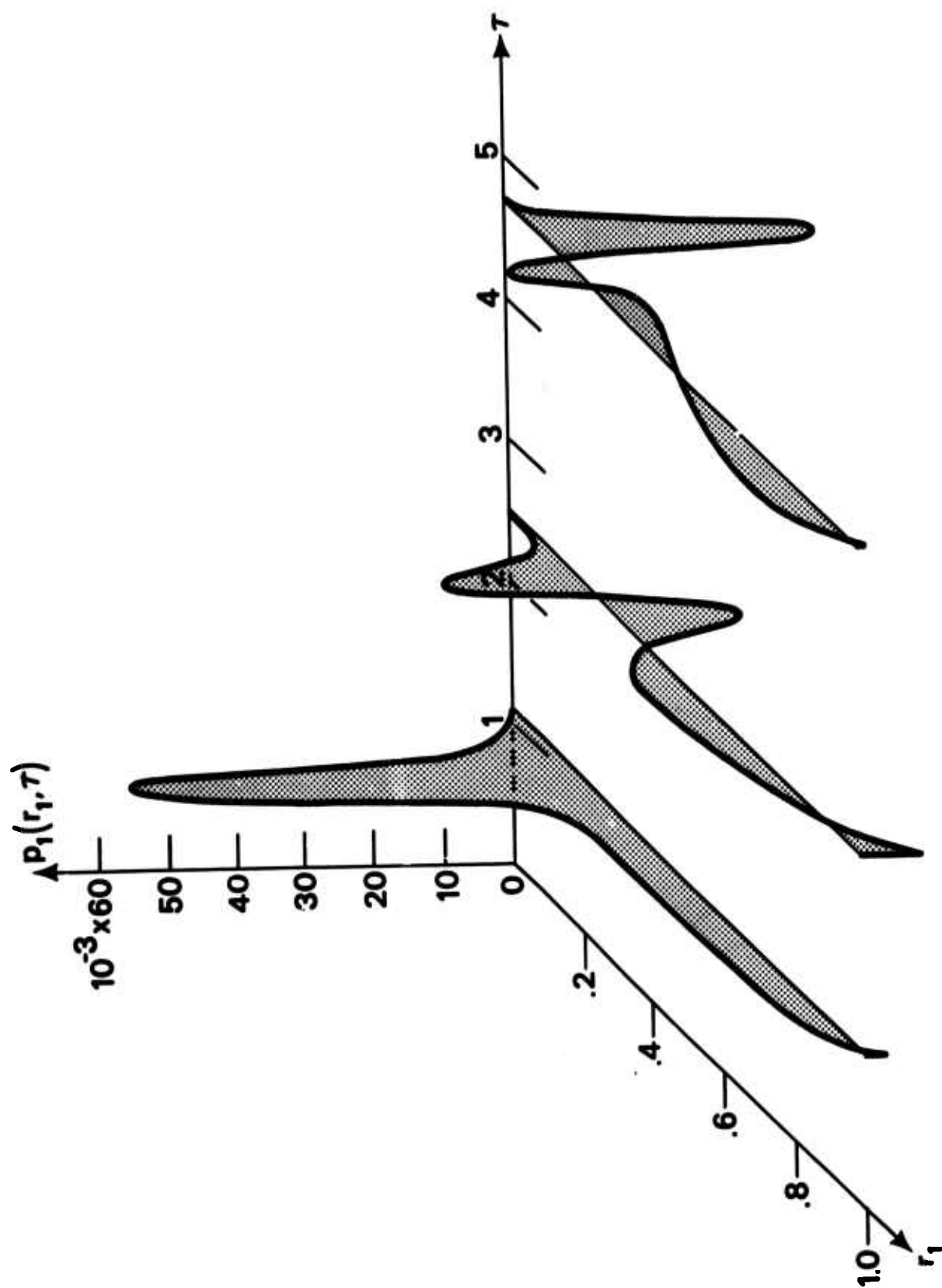
Fig. 2 Nondimensional excess pressure, p_1, vs. nondimensional time, τ, and nondimensional radius, r_1; $\phi=0$ and $\tau_0 =.01\tau_c$, (Elastic shell boundary).
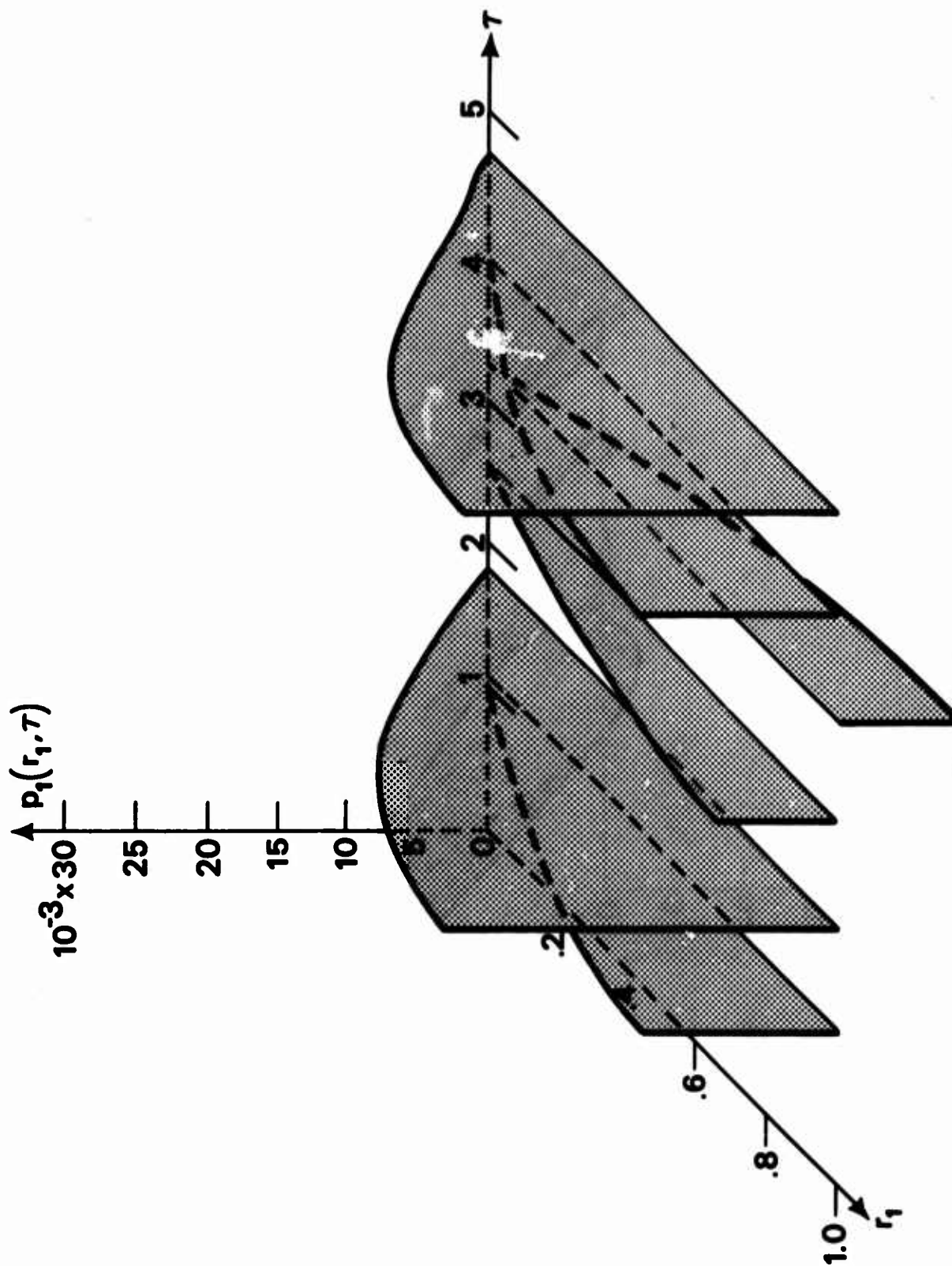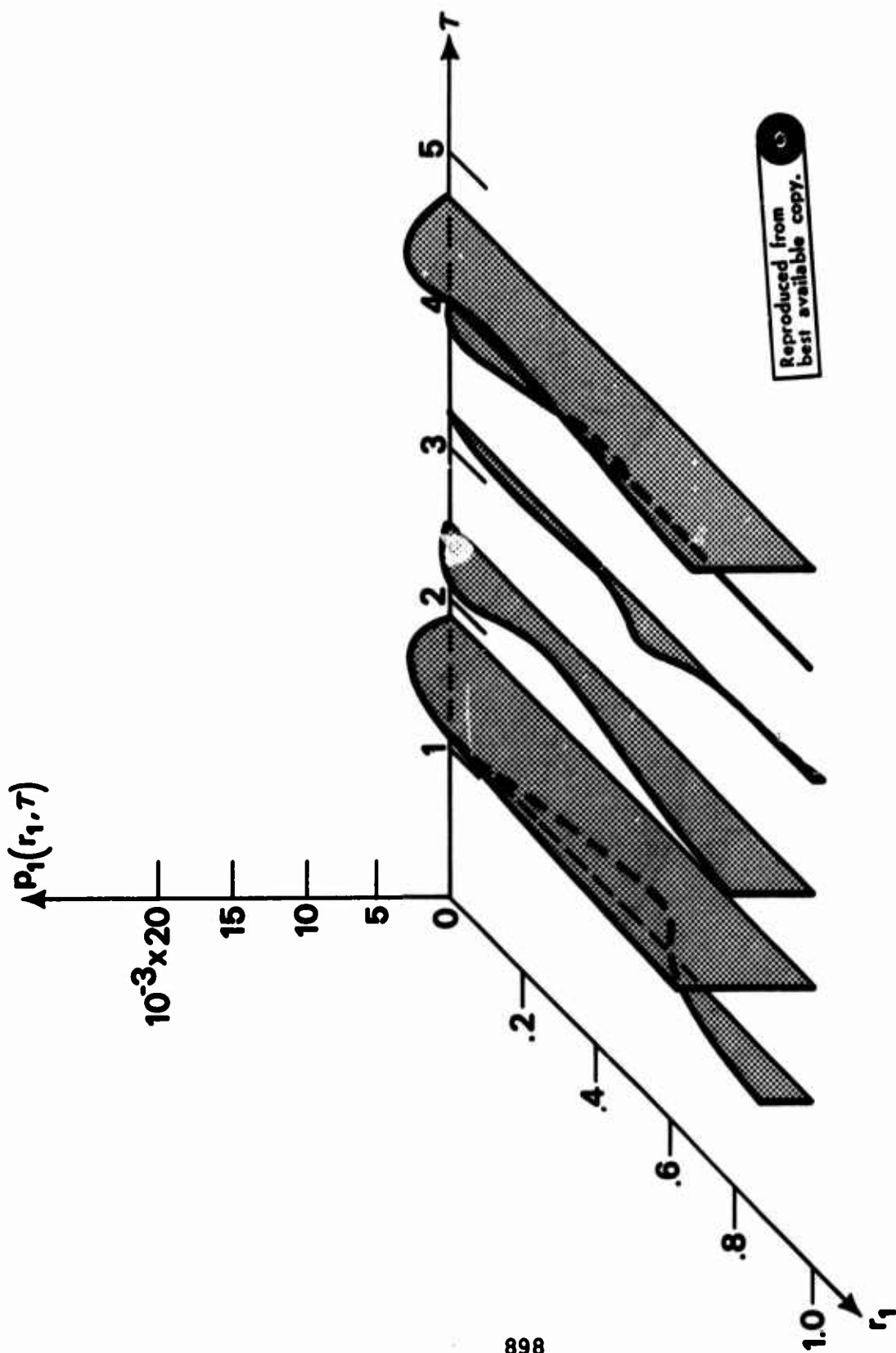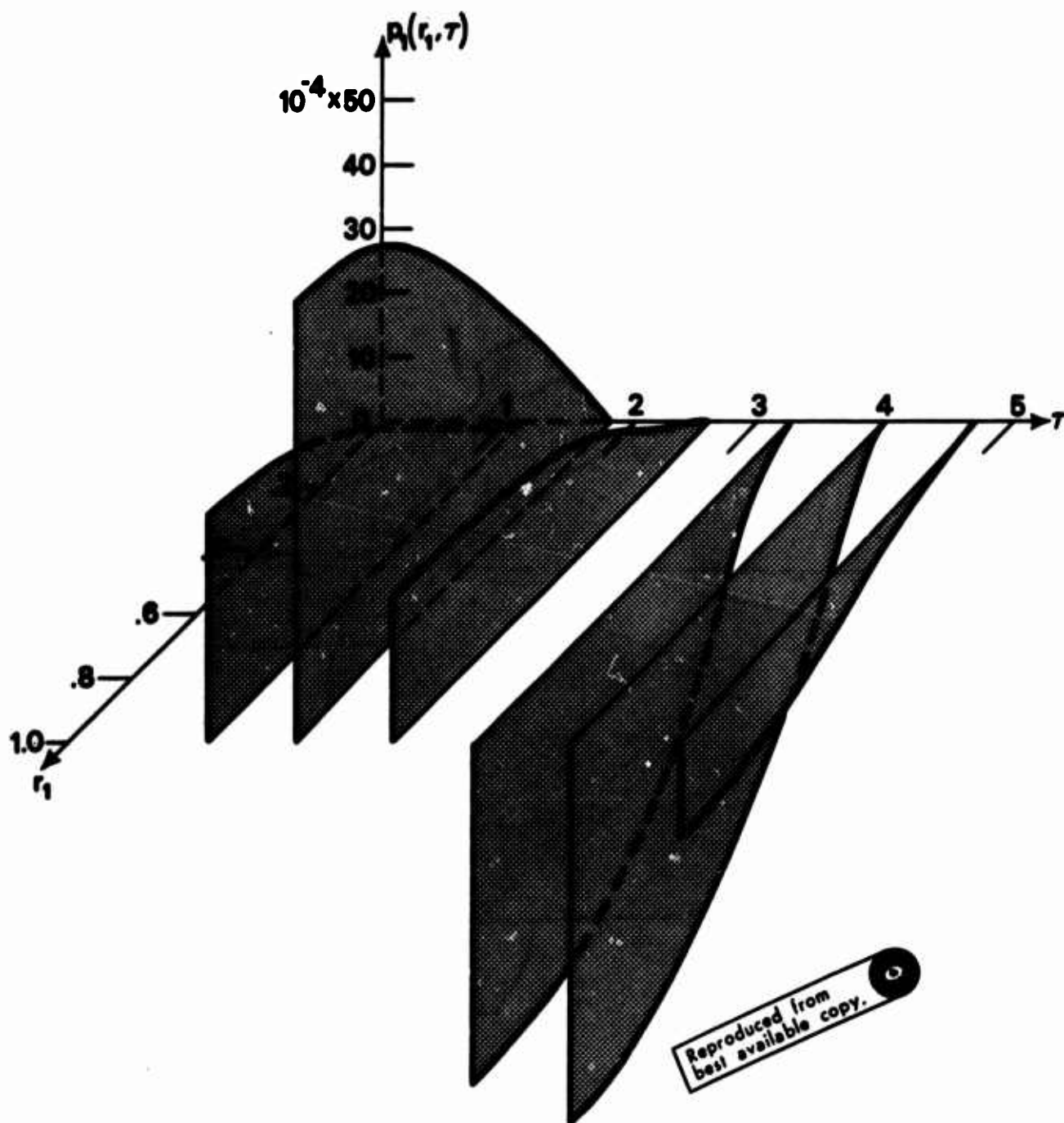
Fig. 3 Nondimensional excess pressure, P_1, vs. nondimensional time, τ, and nondimensional radius, r_1; $\phi=0$ and $\tau_0=0.5\tau_c$. (Rigid shell boundary).
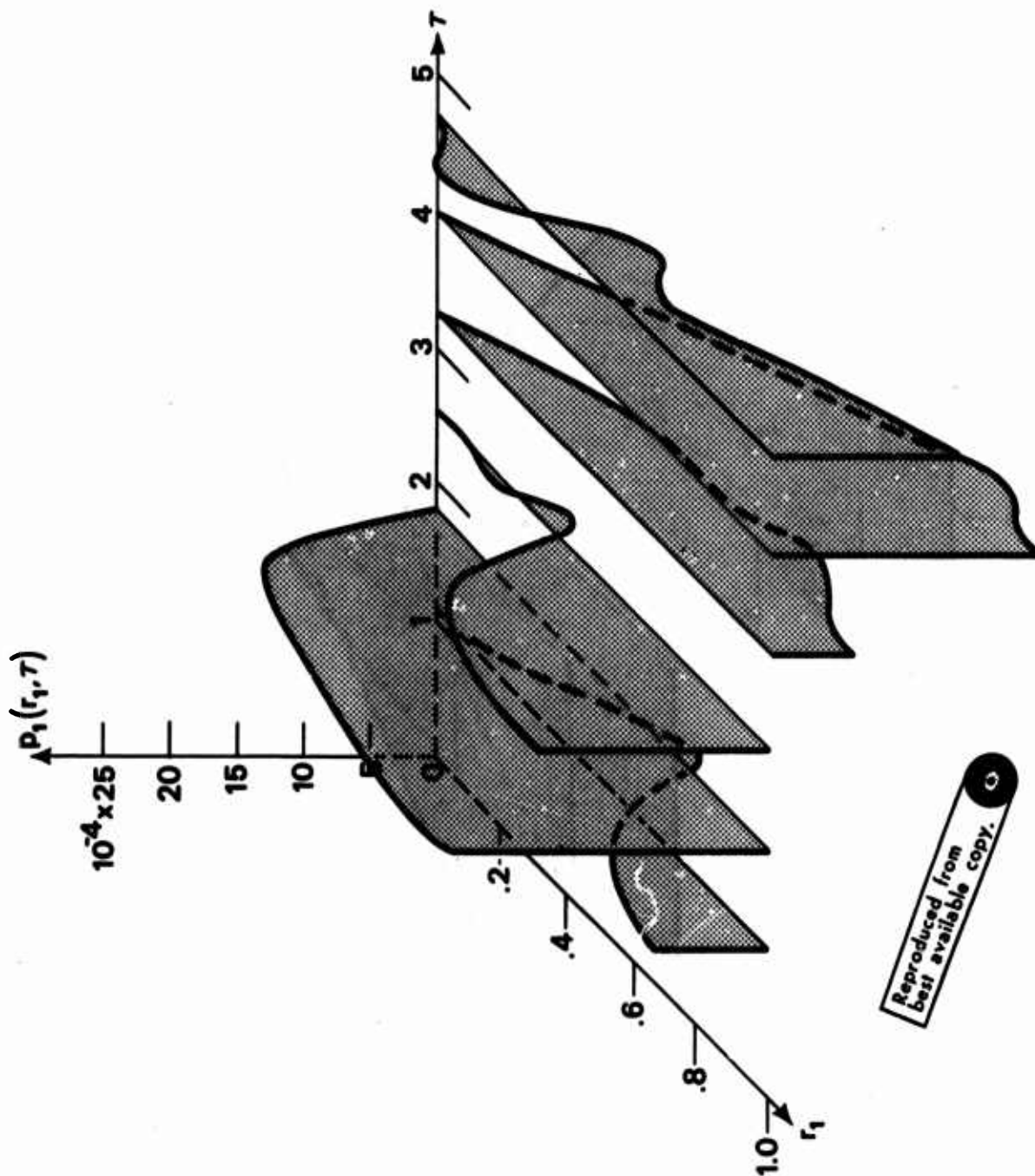
Fig. 4 Nondimensional excess pressure, p_1, vs. nondimensional time, τ, and nondimensional radius, r_1; $\phi=0$ and $\tau_0=0.5\tau_c$. (Elastic shell boundary).

Fig. 5   Nondimensional excess pressure, p_1, vs. nondimensional time, τ, and nondimensional radius, r_1; $\phi = 0$ and $\tau_0 = 2\tau_c$. (Rigid shell boundary).

Fig. 6 Nondimensional excess pressure, p_1, vs. nondimensional time, τ, and nondimensional radius, r_1; $\phi=0$ and $\tau_0=2\tau_c$. (Elastic shell boundary).
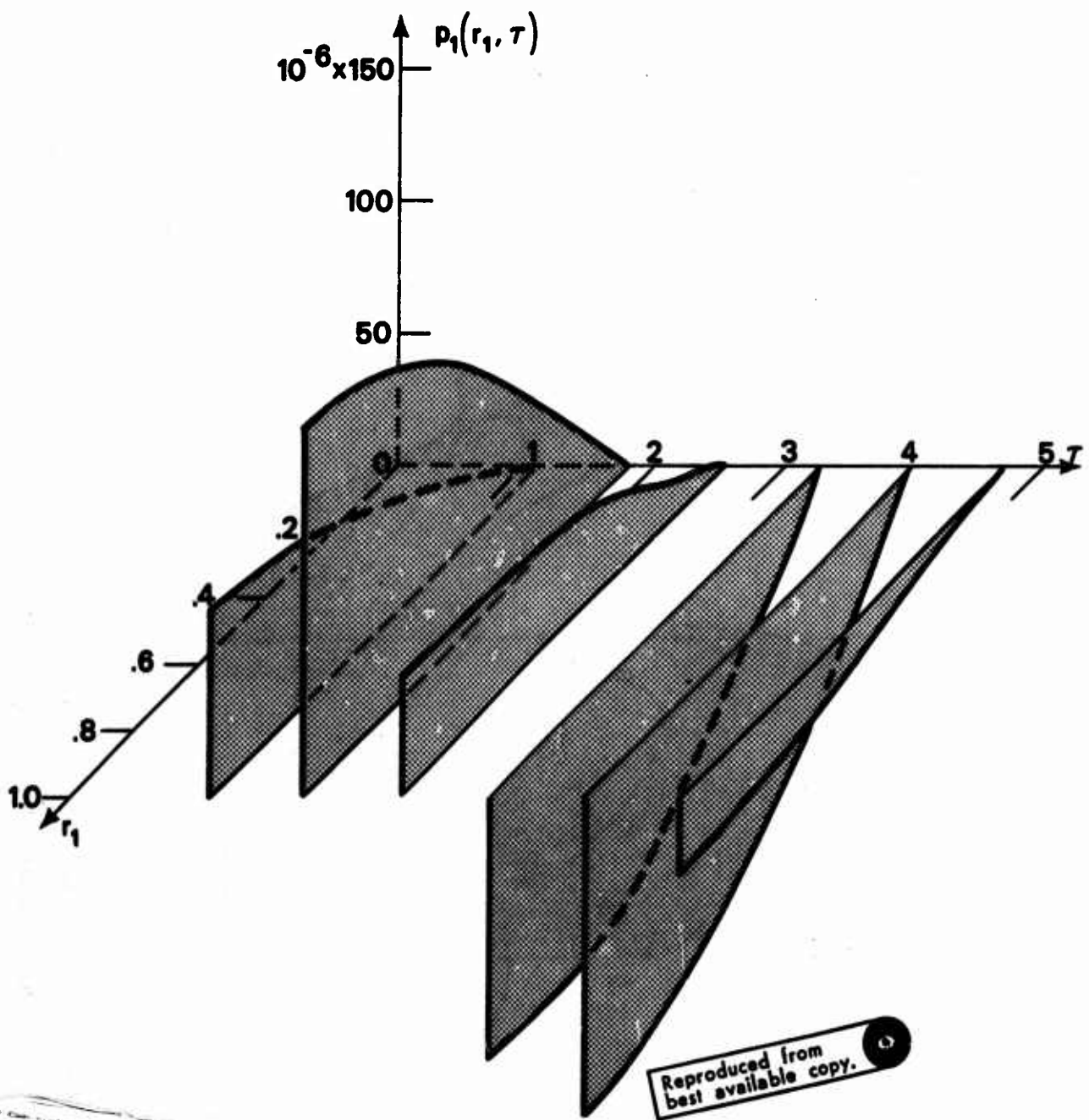
Fig. 7   Nondimensional excess pressure, p_1; vs. nondimensional time, τ, and nondimensional radius, r_1; $\phi = 0$ and $\tau_0 = 100\tau_c$. (Rigid shell boundary).
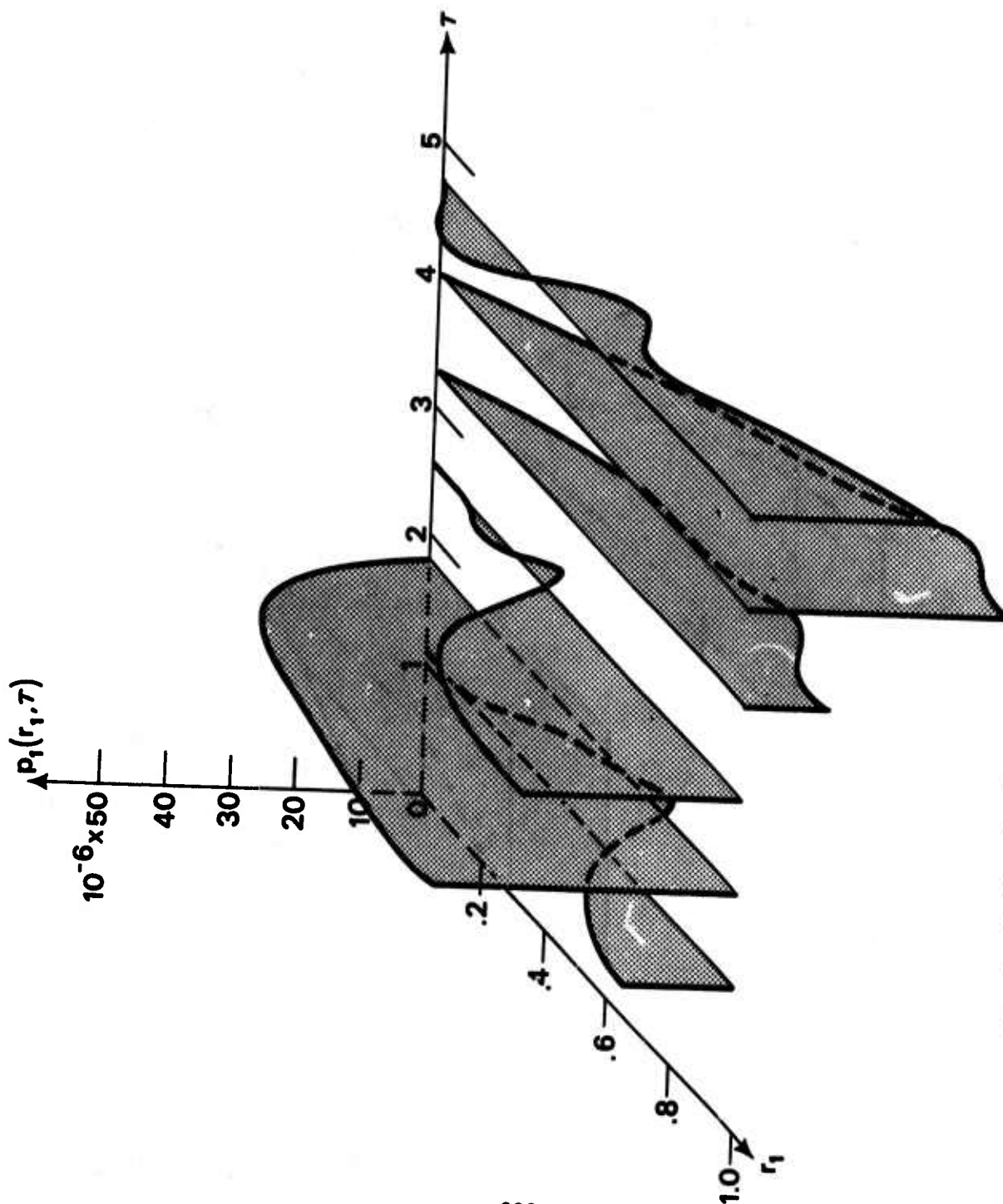
Fig. 8 Nondimensional excess pressure, p_1, vs. nondimensional time, τ, and nondimensional radius, r_1; $\phi=o$ and $\tau_o=100\tau_c$. (Elastic shell boundary).

REFERENCES

1. M. C. Junger, "Vibration of Elastic Shells in a Fluid Medium and Associated Radiation of Sound," J. Appl. Mech. 74, 439-445 (1952)

2. J. E. Greenspon, "Vibration of Thick and Thin Cylindrical Shells Surrounded by Water," J. Acoust. Soc. Amer. 33, 1321-1328 (1961)

3. A. E. Engin and Y.K. Liu, "Axisymmetric Response of a Fluid-Filled Spherical Shell in Free Vibrations" J. Biomech. 3, 11-20 (1970).

4. A. Anzelius, "The Effect of an Impact on a Spherical Liquid Mass" Acta. path. microbiol. scand. Suppl. 48 (1943)

5. W. Güttinger, "Der Stosseffekt auf eine Flüssigkeitskugel als Grundlage einer Physikalischen Theorie der Entstehung von Gehirnverletzungen," Z. Naturforsch. Teil A 5, 622-628 (1950)

6. T. Hayashi, "Study of Intracranial Pressure Caused by Head Impact" 12th Int. Congr. Appl. Mech.

7. A. E. Engin, "The Axisymmetric Response of a Fluid-Filled Spherical Shell to a Local Radial Impulse - A Model for Head Injury" J. Biomech. 2, 325-341 (1969)

8. W. Goldsmith, "The Physical Processes Producing Head Injury" Proceedings of the Head Injury Conference, 350-382, Lippincott, Philadelphia